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ON A CUBIC LIFTING

BY

ZHENGYU MAO*

Department of Mathematics and Computer Science, Rutgers University
Newark, NJ 07102, USA
e-mail: zmao@andromeda.rutgers.edu

AND

STEPHEN RALLIS**

Department of Mathematics, The Ohio State University
Columbus, OH 43210, USA
e-mail: haar@math.ohio-state.edu

ABSTRACT

Let $\tilde{\mathrm{SL}}_2$ be the three-fold cover of SL_2. We prove there is a lifting of the genuine automorphic representations of $\tilde{\mathrm{SL}}_2$ to the automorphic representations of SL_2, using a relative trace formula.

1. Introduction

Let F be a number field, $\mathbb{A}$ its adele ring. We will assume that F contains the cubic roots of 1. Let G be the group SL_2, $\tilde{G}$ be the three fold cover of G. An element in $\tilde{G}$ has the form (g, ζ) with $g \in G$ and $\zeta^3 = 1$. In this paper, we study the cubic lifting from the set of cuspidal genuine representations of $\tilde{G}$ to the set of cuspidal representations of G. The automorphic representations are assumed implicitly to be irreducible.

A cuspidal representation $\pi = \otimes\pi_v$ of G is a cubic lift of a cuspidal genuine representation $\pi' = \otimes\pi'_v$, if for almost all finite places v, $\pi_v = \pi_v(|\cdot|^{s_v})$ and $\pi'_v = \pi'_v(|\cdot|^{s_v/3})$, where s_v is some complex number (§6).

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This cubic lifting is first studied in [G-R-S]. There the authors use the fact that G and $\tilde{G}$ consist a dual pair in an exceptional group of type G_2, i.e. $G \times \tilde{G}$ is a subgroup inside the three fold cover $\tilde{G}_2$ of G_2, with G and $\tilde{G}$ satisfying the double centralizer condition. The cubic lifting is established by restricting the θ-module of $\tilde{G}_2$, constructed in [S], to the subgroup $G \times \tilde{G}$. We now recall their result.

The symmetric cube representation of G: $g \to s^3(g)$ has its image $s^3(G)$ in Sp_2. Fix a nontrivial additive character ψ on $\mathbf{A}/F$. Let ρ_ψ be the Weil representation of $\tilde{Sp}_2$ (double cover of Sp_2) associated to ψ. It acts on $S(\mathbb{A}^2)$ the space of Schwartz functions on $\mathbb{A}^2$. Since $\tilde{Sp}_2$ splits over the subgroup $s^3(G)$, ρ_ψ restricts to a representation of $s^3(G)$. For $\phi \in S(\mathbb{A}^2)$, define for $g \in G(\mathbb{A})$:

$$\Theta_\psi^\phi(s^3(g)) = \sum_{X \in F^2} \rho_\psi(s^3(g))\phi(X). \tag{1}$$

We say a cuspidal representation π of G is Θ-distinguished (relative to ψ) if there exists $f \in \pi$, $\phi \in S(\mathbf{A}^2)$, such that

$$P(\phi, f) = \int_{G(F)\backslash G(\mathbb{A})} \Theta_\psi^\phi(s^3(g)) f(g) dg \tag{2}$$

is nonzero. It is shown in [G-R-S] that a Θ-distinguished cuspidal representation π is a cubic lift of some cuspidal representation π' of $\tilde{G}$.

Our approach to establish the cubic lifting is by using a trace identity. Write $n(t)$ for the element $\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$ in G and $\tilde{n}(t)$ for the element $(n(t), 1)$ in $\tilde{G}$. Let N, $(\tilde{N})$ be the group of elements $n(t)$ (or $\tilde{n}(t)$). Define characters on N and $\tilde{N}$ by $\theta(n(t)) = \psi(-3t)$ and $\theta'(\tilde{n}(t)) = \psi(t)$. Recall for $f \in C_c^\infty(G(\mathbb{A}))$, we have a kernel function

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

With above notations, we define the distribution:

$$I(f, \phi) = \int_{N(F)\backslash N(\mathbb{A})} \int_{G(F)\backslash G(\mathbb{A})} K_f(g, n) \Theta_\psi^\phi(s^3(g)) \theta(n) dg dn. \tag{3}$$

On the other hand, let $f' \in C_c^\infty(\tilde{G}(\mathbb{A}))$ be an anti-genuine function, i.e. $f'(g, \xi) = \xi^{-1} f'(g, 1)$. The group $G(F)$ embeds in $\tilde{G}(\mathbf{A})$ by $g \to (g, 1)$. We may define for $x, y \in \tilde{G}(\mathbf{A})$,

$$K_{f'}(x, y) = \sum_{\gamma \in G(F)} f'(x^{-1} \cdot \gamma \cdot y).$$

One defines the distribution:

$$J(f') = \int_{(\tilde{N}(F)\backslash\tilde{N}(\mathbf{A}))^2} K_{f'}(\tilde{n}_1^{-1}, \tilde{n}_2)\theta'(\tilde{n}_1 \cdot \tilde{n}_2)d\tilde{n}_1 d\tilde{n}_2. \tag{4}$$

Our main result is that for *matching* functions f, ϕ and f',

$$I(f, \phi) = J(f'). \tag{5}$$

Here the concept of matching is as usual defined in terms of conditions over local fields.

The spectral decomposition of $I(f, \phi)$ is roughly a sum over the set of cuspidal representations π of

$$I_\pi(f, \phi) = \sum_{\varphi_i} P(\phi, \pi(f)\varphi_i)\bar{W}_{\varphi_i}(e) \tag{6}$$

where $\{\varphi_i\}$ is an orthonormal basis for π and

$$W_\phi(g) = \int_{N(F)\backslash N(\mathbb{A})} \phi(ng)\theta^{-1}(n)dn \tag{7}$$

is the Whittaker functional; $\bar{W}$ is the complex conjugate of W. Meanwhile, the distribution $J(f')$ is roughly a sum over the set of cuspidal representations π' of certain distributions $J_{\pi'}(f')$. From equation (5), there will be identities between the (sums of) nontrivial distributions $I_\pi(f, \phi)$ and the (sums of) nontrivial distributions $J_{\pi'}(f')$. The cubic lifting follows from these identities.

To be more precise, the cubic lifting obtained from (5) is a correspondence between L-packets of cuspidal representations (§6). An L-packet on G is said to be Θ-distinguished if there is a representation in the packet which is Θ-distinguished. Recall that a CAP representation is a cuspidal representation which is near equivalent to an Eisenstein series ([PS]). Assuming the CAP representations do not exist for $\tilde{G}$, we show there is a bijection between Θ-distinguished L-packets Π on G and *generic* L-packets Π' on $\tilde{G}$, such that each representation $\pi \in \Pi$ is a cubic lift of any representation $\pi' \in \Pi'$. In particular, any cuspidal representation π' of $\tilde{G}$ has a cubic lift to a Θ-distinguished representation on G. This is a result that is missing in [G-R-S]. We note that the definitions of **Θ-distinguished** and **generic** depend on a fixed ψ.

The trace identity (5) fits in a family of trace identities introduced in [M-R]. In [M-R], for a simple Lie group H_1 not of type A, we associate a reductive group H_2, and define a distribution $I(f, \phi)$ on H_2 by integrating the kernel function against a Theta function as in (3) (here one needs an embedding of H_2 in a metaplectic group, see [M-R] for details). The trace identity then compares $I(f,\phi)$ with a distribution $J(f')$ on SL_2 or its covering. We provided some evidence of the trace identities by showing the fundamental lemma. For the case at hand, H_1 is of type G_2. Here we work out the proof for the identity (5) in its entirety. This should be considered as further evidence that the trace identities introduced in [M-R] are valid.

In §2, we collect some preliminaries on the Hecke algebras. In §3, we study the distributions $I(f,\phi)$ and $J(f')$ and their spectral decompositions. The necessary local identities are proved in §4–§5. We establish the cubic lifting in §6.

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2. Hecke algebras

In this section, let F_v be a local nonarchimedean field, R_v be its integer ring, P_v be its prime ideal, ϖ_v be a uniformizer in P_v. Assume that $q_v = |R_v/P_v|$ is odd and equals 1 modulo 3. Since F contains the cubic roots of 1, for almost all places v, q_v satisfies this condition. In this section, we will drop the reference to the place v. We will denote by d_a the diagonal matrix $\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}$.

We establish an algebra isomorphism between the Hecke algebras of G and $\tilde{G}$.

(2.1) Let T be the group of diagonal matrices in G. Denote by $\rho(\chi)$ the representation of $G(F)$ induced from the character of TN:

$$d_a n(x) \to \chi(a).$$

Then $\rho(\chi)$ contains a vector fixed under $K = G(2,R)$ if and only if χ is unramified; call φ such a K-fixed vector. Assume $\chi(\varpi) = q^s$ with s a complex number. For f in the Hecke algebra H of bi-K-invariant compactly supported functions on $G(F)$, we have

$$\rho(\chi)(f)\varphi = f^\wedge(s)\varphi,$$

and $f \to f^\wedge(s)$ is an algebra homomorphism of H into $\mathbf{C}$ (Satake transform). Any homomorphism of H into $\mathbf{C}$ has this form for some s.

Let f_m be the characteristic function of $Kd_{\varpi^m}K$. Then $\{f_m | m \geq 0\}$ is a basis of H. Using the Iwasawa decomposition, we get

$$Kd_{\varpi^m}K = \bigsqcup_{i=-m,\dots,m} n(z)d_{\varpi^i}K \tag{8}$$

with z going through the cosets $\varpi^{i-m}(R^\times/P^{m+i})$ when $i \neq m, -m$, and $z \in R/P^{2m}$ when $i = m$, $z = 0$ when $i = -m$. It is then easy to see that

$$f_0^\wedge(s) \equiv 1 \tag{9}$$

and, when $m \geq 1$,

$$f_m^\wedge(s) = q^m(q^{ms} + q^{-ms}) + q^m(1 - q^{-1}) \sum_{i=1-m}^{m-1} q^{is}. \tag{10}$$

(2.2) Let $\tilde{G} = \tilde{SL}_2$ be the three fold cover of $SL(2)$. Any element in $\tilde{G}$ can be written as (g, ξ) with $g \in G$ and $\xi \in \mu_3$ the set of cubic roots of unity. Denote by (a, b) the cubic Hilbert symbol. Then the multiplicative law over $\tilde{G}$ is

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \alpha(g_1, g_2))$$

where α is certain cocycle we now describe. For

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

set $x(g) = c$ if $c \neq 0$ and $x(g) = d$ if $c = 0$. Then

$$\alpha(g_1, g_2) = \left(\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)}\right).$$

With the assumption on F at the beginning of the section, α is cohomologically trivial on $K = G(R)$. It is more convenient to use an equivalent cocycle β identically one on $K \times K$. For g given as in the above, define $s(g) = (c, d)$ if $cd \neq 0$ and the valuation of c does not divide 3; let $s(g) = 1$ otherwise. Set

$$\beta(g_1, g_2) = \alpha(g_1, g_2) s(g_1) s(g_2) s(g_1 g_2)^{-1}$$

and define the new multiplicative law over the set of $\tilde{G}$:

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \beta(g_1, g_2)).$$

The map $k \to (k, 1)$ is then a homomorphism from K to $\tilde{G}$; we will again denote by K its image. The Hecke algebra H' on $\tilde{G}$ is the set of bi-K-invariant compactly supported antigenuine functions on $\tilde{G}$.

(2.3) The subgroup $\tilde{T}^3$ consisting of elements (d_{a^3}, ξ) is a maximal abelian subgroup in $\tilde{G}$. Denote by $\tilde{\rho}(\chi)$ the representation of $\tilde{G}(F)$ induced from the character on $\tilde{T}^3 \cdot \tilde{N}$:

$$(d_{a^3} \cdot n(x), \xi) \to \xi \chi(a^3).$$

The representation contains a K-fixed vector, necessarily with multiplicity 1, if and only if χ^3 is unramified [Ar]. Call φ such a vector; assume $\chi^3(\varpi) = q^{3s}$ for s a complex number. Then

$$\tilde{\rho}(\chi)(f')\varphi = f'^{\wedge}(s)\varphi$$

for any $f' \in H'$, and $f' \to f'^{\wedge}(s)$ is a homomorphism from H' to $\mathbf{C}$. Any homomorphism from H' to $\mathbf{C}$ takes this form.

Let f'_m be an anti-genuine function of $\tilde{G}$ defined as follows: if $(g, \xi') \in \tilde{G}$ is of the form

$$k_1 \cdot (d_{\varpi^{3m}}, \xi) \cdot k_2, k_1, k_2 \in K,$$

then $f'_m(g, \xi') = \xi^{-1}$; the function equals 0 otherwise. The set $\{f'_m | m \geq 0\}$ is a basis of H'. We now determine $f'^{\wedge}_m(s)$. Recall

$$\tilde{\rho}(\chi)(f'_m)\varphi = \int_{g \in K \cdot (d_{\varpi^{3m}}, 1) \cdot K} \tilde{\rho}(\chi)(g)\varphi dg. \tag{11}$$

Similar to (8), we have a decomposition:

$$K \cdot (d_{\varpi^{3m}}, 1) \cdot K = \bigsqcup_{i=-3m,\ldots,3m} (n(z), \xi(z)) \cdot d_{\varpi^i} \cdot K \tag{12}$$

when $i \neq 3m, -3m$, z goes through the cosets $\varpi^{i-3m}(R^\times / P^{3m+i})$ with $\xi(z) = (\varpi^i, z)$; when $i = 3m$, $z \in R/P^{6m}$ and $\xi(z) \equiv 1$; when $i = -3m$, $z = 0$ and $\xi(z) = 1$. When i is not divisible by 3, $\xi(z) \not\equiv 1$, the contribution from the set $(n(z), \xi(z)) \cdot d_{\varpi^i} \cdot K$ to the integral (11) is 0. When i is divisible by 3, $\xi(z) \equiv 1$; thus from (11) we have

$$f'^{\wedge}_0(s) \equiv 1 \tag{13}$$

and, when $m \geq 1$,

$$f'^{\wedge}_m(s) = q^{3m}(q^{3ms} + q^{-3ms}) + q^{3m}(1 - q^{-1}) \sum_{i=1-m}^{m-1} q^{3is}. \tag{14}$$

(2.4) We have proved:

Proposition 1: *For s any complex number, $f^{\wedge}_m(s) = q^{-2m} f'^{\wedge}_m(s/3)$. There is an algebra isomorphism from H to H' given by $f_m \to q^{-2m} f'_m$.*

3. The distributions $I(f,\phi)$ and $J(f')$

(3.1) Recall

$$I(f,\phi)=\int_{N(F)\backslash N(\mathbf{A})}\int_{G(F)\backslash G(\mathbf{A})}K_f(g,n)\theta(n)\Theta_\psi^\phi(s^3(g))dgdn \tag{15}$$

with

$$K_f(x,y)=\sum_{\gamma\in G(F)}f(x^{-1}\gamma y)\quad\text{and}\quad\Theta_\psi^\phi(s^3(g))=\sum_{(a,b)\in F^2}\rho_\psi(s^3(g))\phi(a,b).$$

In [M-R], we consider the unwinding of the integral $I(f,\phi)$. Let $f=\otimes f_v\in C_c^\infty(G(\mathbf{A}))$, $\phi=\otimes\phi_v\in S(\mathbf{A}^2)$. Define

$$f_v*\phi_v(x,y)=\int_{G(F_v)}f_v(g^{-1})\rho_\psi(s^3(g))\phi_v(x,y)dg. \tag{16}$$

The following proposition expresses $I(f,\phi)$ in terms of local orbital integrals:

Proposition 2:

$$I(f,\phi)=\sum_{a\in F^\times}\prod_v I_v(a_v,f_v*\phi_v)+\prod_v I_v^+(f_v*\phi_v)+\prod_v I_v^-(f_v*\phi_v) \tag{17}$$

where

$$I_v(a,f_v*\phi_v)=\int_{F_v}f_v*\phi_v(a,t)\psi\Big(\frac{t^3-3t}{a}\Big)dt, \tag{18}$$

$$I_v^\pm(f_v*\phi_v)=f_v*\phi_v(0,\pm1). \tag{19}$$

For completeness, we include a *formal* proof of the proposition. As $\Theta_\psi^\phi(s^3(g))$ is left $G(F)$-invariant, the integral (15) is

$$\int_{N(F)\backslash N(\mathbf{A})}\int_{G(F)\backslash G(\mathbf{A})}\sum_{\gamma\in G(F)}f(g^{-1}\gamma n)\theta(n)\Theta_\psi^\phi(s^3(g))dgdn$$
$$=\int_{G(\mathbf{A})\times N(F)\backslash N(\mathbf{A})}f(g^{-1}n)\theta(n)\Theta_\psi^\phi(s^3(g))dgdn.$$

Change $g\to ng$:

$$I(f,\phi)=\int_{G(\mathbf{A})\times N(F)\backslash N(\mathbf{A})}f(g^{-1})\theta(n)\Theta_\psi^\phi(s^3(n)s^3(g))dgdn$$
$$=\int_{N(F)\backslash N(\mathbf{A})}\Theta_\psi^{f*\phi}(s^3(n))\theta(n)dn.$$

Use the explicit formula for $\rho_\psi(s^3(n(t)))$ ([G-R-S]); we get

(20) $$I(f,\phi) = \int_{\mathbf{A}/F} \sum_{(a,b)\in F^2} f*\phi(a, b+at)\psi(a^2t^3+3abt^2+3b^2t-3t)dt.$$

First consider the contribution to the above sum from the part $a=0$, denoted $I_s(f*\phi)$. Then

$$I_s(f*\phi) = \int_{\mathbf{A}/F} \sum_{b\in F} f*\phi(0,b)\psi(3b^2t-3t)dt.$$

The integral over t is nonzero only when $b=\pm 1$; as the integral over t then gives a factor of 1, we get

$$I_s(f*\phi) = \prod_v I_v^+(f_v*\phi_v) + \prod_v I_v^-(f_v*\phi_v).$$

Denote by $I(a, f*\phi)$ the contribution to the sum in (20) from each $a \in F^\times$. Make a change of variable $t \to \frac{t-b}{a}$; we get

$$I(a, f*\phi) = \int_{\mathbf{A}/F} \sum_{b\in F} f*\phi(a,t)\psi\Big(\frac{t^3-3t}{a}\Big)\psi\Big(\frac{-b^3+3b}{a}\Big)dt.$$

Since $\psi(\dfrac{-b^3+3b}{a}) = 1$,

$$I(a, f*\phi) = \int_{\mathbf{A}} f*\phi(a,t)\psi\Big(\frac{t^3-3t}{a}\Big)dt.$$

Thus the proposition.

(3.2) We now consider the spectral decomposition of $I(f,\phi)$. Our goal is to show Proposition 3 at the end of this subsection.

At each finite place v, set $K_v = G(R_v)$ as in §2. At an infinite place v, let K_v be the unitary group in G. Let $K = \prod K_v$. For each idele class character χ, let $V(\chi)$ be the space of functions φ on K such that

$$\varphi(d_a n(x)k) = \chi(a)\varphi(k), \quad k\in K,\ d_a n(x) \in K.$$

For each $s \in \mathbf{C}$, one may identify $V(\chi)$ with a space of functions on $G(\mathbf{A})$ by extending a $\varphi \in V(\chi)$ to a function $\varphi(g,s)$ on $G(\mathbf{A})$, with

$$\varphi(d_a n(x)k, s) = \chi(a)|a|^{s+1}\varphi(k), \quad k\in K.$$

The group $G(\mathbf{A})$ acts on $V(\chi)$ by right shift. We get a representation denoted as $\rho_s(\chi)$.

From [L], we have

$$K_f(x,y) = \sum_{\pi} K_{\pi,f}(x,y) + \sum_{\chi} K_{\chi,f}(x,y) \tag{21}$$

where π is either a cuspidal representation or the trivial representation, χ is an idele class character and

$$K_{\pi,f}(x,y) = \sum_{\varphi_i} \pi(f)\varphi_i(x)\bar{\varphi}_i(y), \tag{22}$$

$$K_{\chi,f}(x,y) = \frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} \sum_{\varphi_i} E(x,\rho_s(\chi)(f)\varphi_i,s)\bar{E}(y,\varphi_i,s)ds. \tag{23}$$

The sum is taken over the orthonormal basis of π or $V(\chi)$. The function $E(x,\varphi,s)$ is the Eisenstein series:

$$E(g,\varphi,s) = \sum_{\gamma\in TN(F)\backslash G(F)} \varphi(\gamma g,s).$$

The Eisenstein series is defined for $\mathrm{Re} s > 1$ and extended meromorphically to the whole complex plane.

Assume f is a K-finite function for the moment. From proposition 2.1 in [J], over the domain of integration for (15), for any N, there exists a c such that for any g in the Siegel domain

$$\sum_{\pi} |K_{\pi,f}(g,n)| + \sum_{\chi} |K_{\chi,f}(g,n)| \leq c|\ g|\ ^{-N}. \tag{24}$$

As $\Theta_{\psi}^{\phi}(s^3(g))$ is a moderately increasing function in g, the integration in (15) and the sum in (21) are interchangeable:

$$I(f,\phi) = \sum_{\pi} I_{\pi}(f,\phi) + \sum_{\chi} I_{\chi}(f,\phi) \tag{25}$$

with the distributions

$$I_{\pi}(f,\phi) = \int_{N(F)\backslash N(\mathbf{A})}\int_{G(F)\backslash G(\mathbf{A})} K_{\pi,f}(g,n)\theta(n)\Theta_{\psi}^{\phi}(s^3(g))dgdn, \tag{26}$$

$$I_{\chi}(f,\phi) = \int_{N(F)\backslash N(\mathbf{A})}\int_{G(F)\backslash G(\mathbf{A})} K_{\chi,f}(g,n)\theta(n)\Theta_{\psi}^{\phi}(s^3(g))dgdn. \tag{27}$$

Moreover, the sum (25) is absolutely convergent.

One easily verifies $I_\pi(f,\phi)\equiv 0$ if π is the trivial representation. For π being any cuspidal representation, with the notations in the introduction,

$$I_\pi(f,\phi)=\sum_{\varphi_i} P(\phi,\pi(f)\varphi_i)\overline{W_{\varphi_i}}(e). \tag{28}$$

We now consider $I_\chi(f,\phi)$. One needs to use the truncation operator. For $h(g)$ a function on $G(\mathbf{A})$, define

$$\wedge^T h(g)=h(g)-\sum_{\gamma\in TN(F)\backslash G(F)} h_N(\gamma g)\chi_T(H(\gamma g)),$$

where $H(g)$ is the height function such that $H(n(x)d_a k)=|a|$ if $k\in K$; χ_T is the characteristic function of the interval (T,∞); and

$$h_N(g)=\int_{N(\mathbf{A})} h(ng)dn.$$

From proposition 2.1 in [J], we know over the domain of integration in (27), when T is sufficiently large, $K_{\chi,f}(g,n)=\wedge_1^T K_{\chi,f}(g,n)$ the truncated function with respect to the first variable. Thus $I_\chi(f,\phi)$ equals

$$\lim_{T\to\infty}\int_{N(F)\backslash N(\mathbf{A})}\int_{G(F)\backslash G(\mathbf{A})}\int_{-i\infty}^{+i\infty}\sum_{\varphi_i}\frac{1}{4\pi i}\wedge^T E(g,\rho_s(\chi)(f)\varphi_i,s)$$
$$\times\bar{E}(n,\varphi_i,s)\theta(n)\Theta_\psi^\phi(s^3(g))dsdgdn.$$

It follows from equation (18) in [J] that for fixed T the above integral is absolutely convergent. In particular, we may change the order of integration. Denote by φ_i' the function $\rho_s(\chi)(f)\varphi_i$. The distribution $I_\chi(f,\phi)$ equals

$$\lim_{T\to\infty}\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty}\sum_{\varphi_i}\bar{W}(e,\varphi_i,s)\int_{G(F)\backslash G(\mathbf{A})}\wedge^T E(g,\varphi_i',s)\Theta_\psi^\phi(s^3(g))dgds \tag{29}$$

where

$$W(g,\varphi,s)=\int_{N(F)\backslash N(\mathbf{A})} E(ng,\varphi,s)\theta^{-1}(n)dn.$$

We claim that (29) equals

$$\frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} I_{\chi,s}(f,\phi)ds+\delta(\chi)I_1'(f,\phi), \tag{30}$$

where $I_{\chi,s}$ and I_1' are some distributions, $\delta(\chi)=1$ when χ is trivial and 0 otherwise; the integral in (30) is absolutely convergent.

Proof of the claim: The integral (29) has the form

$$\lim_{T\to\infty}\int_{-i\infty}^{+i\infty} H(s,T)ds.$$

In the notation of $H(s,T)$, we implicitly have the dependence on f,ϕ and χ. From the above discussion, $H(s,T)$ is a meromorphic in s, and the above integral is absolutely convergent. To write the integral in the form of (30), we will separate $H(s,T)$ into a sum of four functions, which we now describe.

Let $M(s,\chi)$ be the intertwining operator from $V(\chi)$ to $V(\chi^{-1})$; then when $\mathrm{Re} s$ is large enough, $\wedge^T E(g,\varphi,s)$ equals ([J-L])

$$\sum_{\gamma\in TN(F)\backslash G(F)} [\varphi(\gamma g,s)(1-\chi_T(H(\gamma g)))-(M(s,\chi))\varphi(\gamma g,-s)\chi_T(H(\gamma g))].$$

The integration over g in (29) equals

$$\text{(31)}\qquad \int_{TN(F)\backslash G(\mathbf{A})} \varphi_i'(g,s)(1-\chi_T(H(g)))\Theta_\psi^\phi(s^3(g))dg$$

$$\text{(32)}\qquad -\int_{TN(F)\backslash G(\mathbf{A})} M(s,\chi)\varphi_i'(g,-s)\chi_T(H(g))\Theta_\psi^\phi(s^3(g))dg.$$

From [G-R-S], we see that

$$\int_{N(F)\backslash N(\mathbf{A})}\Theta_\psi^\phi(s^3(ng))dn=\rho_\psi(s^3(g))\phi(0,0)+\sum_{\xi\in F^\times}\int_{\mathbf{A}}\rho_\psi(s^3(g))\phi(\xi,t)\psi\Big(\frac{t^3}{\xi}\Big)dt.$$

From this identity and the Iwasawa decomposition, we get

$$4\pi i H(s,T)=H_1(s,T)+H_2(s,T)+H_3(s,T)+H_4(s,T),$$

where

$$H_1(s,T)=\sum_{\varphi_i}\bar W(e,\varphi_i,s)\int_0^T\int_{K(\mathbf{A})}|a|^{s-1}\varphi_i'(k)\rho_\psi(s^3(d_ak))\phi(0,0)dkd^\times a,$$

$$H_2(s,T)=\sum_{\varphi_i}\bar W(e,\varphi_i,s)\int_0^T\int_{K(\mathbf{A})}|a|^{s-1}\varphi_i'(k)$$
$$\times\sum_{\xi\in F^\times}\int_{\mathbf{A}}\rho_\psi(s^3(d_ak))\phi(\xi,t)\psi\Big(\frac{t^3}{\xi}\Big)dtdkd^\times a,$$

$$H_3(s,T) = -\sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_T^\infty \int_{K(\mathbf{A})} |a|^{-s-1}[M(s,\chi)\varphi_i'](k) \times \rho_\psi(s^3(d_a k))\phi(0,0)dkd^\times a,$$

$$H_4(s,T) = -\sum_{\varphi_i} \bar{W}(e,\varphi_i,s) \int_T^\infty \int_{K(\mathbf{A})} |a|^{-s-1}[M(s,\chi)\varphi_i'](k) \times \sum_{\xi\in F^\times} \int_{\mathbf{A}} \rho_\psi(s^3(d_a k))\phi(\xi,t)\psi\Big(\frac{t^3}{\xi}\Big)dtdkd^\times a.$$

First consider $H_1(s,T)$. Note by our notation, $\varphi_i'(k) = \rho_s(\chi)(f)\varphi_i(k)$. Thus $H_1(s,T)$ equals

$$\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\rho_s(\chi)(f)\varphi_i *_K \phi(0,0) \int_0^T |a|^{-2}|a|^{s+1}|a|^2 d^\times a$$

where

$$\varphi *_K \phi(x,y) = \int_{K(\mathbf{A})} \varphi(k)\rho_\psi(s^3(k))\phi(x,y)dk. \tag{33}$$

Consider the expression $\varphi *_K \phi(0,0)$ with $\varphi = \rho_s(\chi)(f)\varphi_i$. Make a change of variable $k \to d_a k$ in (33), where $|a| = 1$; it equals

$$\int_{K(\mathbf{A})} \rho_s(\chi)(f)\varphi_i(d_a k)\rho_\psi(s^3(d_a k))\phi(0,0)dk$$
$$=\chi(a)\int_{K(\mathbf{A})} \rho_s(\chi)(f)\varphi_i(k)\rho_\psi(s^3(k))\phi(0,0)dk,$$

which is $\chi(a)\rho_s(\chi)(f)\varphi_i *_K \phi(0,0)$. Thus $\rho_s(\chi)(f)\varphi_i *_K \phi(0,0)$ is nonzero only when χ is trivial. Assume $\chi \equiv 1$, $H_1(s,T)$ equals

$$\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\frac{T^{s+1}}{s+1}\rho_s(1)(f)\varphi_i *_K \phi(0,0).$$

Similarly, $H_3(s,T)$ is nonzero only when $\chi \equiv 1$, when it equals

$$\sum_{\varphi_i} \bar{W}(e,\varphi_i,s)\frac{T^{1-s}}{1-s}[M(s,1)\rho_s(1)(f)\varphi_i] *_K \phi(0,0).$$

Thus $H_1(s,T) + H_3(s,T)$ is meromorphic in s, it equals 0 unless $\chi \equiv 1$. When $\chi \equiv 1$, let $I_1'(f,\phi)$ be

$$\lim_{T\to\infty} \int_{-i\infty}^{+i\infty} \frac{1}{4\pi i}(H_1(s,T) + H_3(s,T))ds.$$

It equals

$$\lim_{T\to\infty}\int_{-i\infty}^{+i\infty}\frac{1}{4\pi i}\sum_{\varphi_i}\Big[\frac{T^{s+1}}{s+1}\rho_s(1)(f)\varphi_i *_K \phi(0,0) \\ + \frac{T^{1-s}}{1-s}(M(s,1)\rho_s(1)(f)\varphi_i) *_K \phi(0,0)\Big]\bar{W}(e,\varphi_i,s)ds.$$

Observe that when s is purely imaginary, the action of $M(s,1)$ on a function of irreducible K-type in $V(1)$ is a scalar multiplication, and it is a unitary action. Use the functional equation of Eisenstein series, we see that the above integral equals

$$\lim_{T\to\infty}\int_{-i\infty}^{+i\infty}\frac{1}{2\pi i}\sum_{\varphi_i}\frac{T^{1+s}}{1+s}\rho_s(1)(f)\varphi_i *_K \phi(0,0)\bar{W}(e,\varphi_i,s)ds.$$

Change s to $-s$, then shift the integration to a line $\mathrm{Re}s = s_0 > 1$, and then take the limit; we get

$$I_1'(f,\phi) = \sum_{\varphi_i}\rho_{-1}(1)(f)\varphi_i *_K \phi(0,0)\bar{W}(e,\varphi_i,-1). \tag{34}$$

We now consider $H_4(s,T)$. Using the notation of (33), it equals

$$-\sum_{\varphi_i}\bar{W}(e,\varphi_i,s)\int_T^{\infty}|a|^{-s+1}\sum_{\xi\in F^\times}\int_{\mathbf{A}}[M(s,\chi)\varphi_i'] *_K \phi(a^3\xi,at)\psi\Big(\frac{t^3}{\xi}\Big)dtd^\times a.$$

For any $N>0$, there is a constant $c(s,\varphi',\phi,N)$ such that

$$|[M(s,\chi)\varphi_i'] *_K \phi(a^3\xi,at)| \le c(s,\varphi',\phi,N)|at\xi_\infty|^{-N}.$$

From this bound, it is easy to see that $H_4(s,T)$ is meromorphic in s and has an upper bound that is independent of T (if $T>1$). On the imaginary line, $H_4(s,T)$ is holomorphic and rapidly decreasing; the integral $\int_{-i\infty}^{+i\infty}|H_4(s,T)|ds$ converges, and is bounded by a constant independent of T.

For the function $H_2(s,T)$, we separate the integration over a into two parts, one from 0 to a fixed $T_0>1$, the other from T_0 to T. Call these two parts $H_{2,1}(s,T_0)$ and $H_{2,2}(s,T,T_0)$ respectively. Similar to the argument for $H_4(s,T)$, we see $H_{2,2}(s,T,T_0)$ is meromorphic in s, and the integral $\int_{-i\infty}^{+i\infty}|H_{2,2}(s,T,T_0)|ds$ converges and is bounded by a constant independent of T. Since H, H_1, H_3, H_4 and $H_{2,2}$ are meromorphic in s, so is $H_{2,1}$, and $\int_{-i\infty}^{+i\infty}|H_{2,1}(s,T_0)|ds$ converges, and is clearly bounded by a constant independent of T.

We conclude that $H_2(s,T) + H_4(s,T)$ is meromorphic in s, and

$$\int_{-i\infty}^{+i\infty} |H_2(s,T) + H_4(s,T)|ds$$

converges and is bounded by a constant independent of T. Thus

$$\lim_{T\to\infty} \int_{-i\infty}^{+i\infty} [H_2(s,T) + H_4(s,T)]ds = \int_{-i\infty}^{+i\infty} \lim_{T\to\infty} [H_2(s,T) + H_4(s,T)]ds$$

where the integration is absolutely convergent. Let

$$I_{\chi,s}(f,\phi) = \lim_{T\to\infty} [H_2(s,T) + H_4(s,T)].$$

We then get identity (30) and our claim. ∎

We note when χ is nontrivial,

$$I_{\chi,s}(f,\phi) = 4\pi i \lim_{T\to\infty} H(s,T). \tag{35}$$

We have now obtained the spectral decomposition for $I(f,\phi)$.

Proposition 3: *With the above notations, for $f \in C_c^\infty(G(\mathbf{A}))$*

$$I(f,\phi) = \sum_\pi I_\pi(f,\phi) + \sum_\chi \int_{-i\infty}^{+i\infty} \frac{1}{4\pi i} I_{s,\chi}(f,\phi)ds + I_1'(f,\phi). \tag{36}$$

The sum of π is over all cuspidal representations; the sum of χ is over all idele class characters. The sum and integral in (36) are absolutely convergent.

Proof: When f is K-finite, the above argument gives (36). The statement on absolute convergence follows from the absolute convergence of (25), (30), the equation (35) and the bound [J](18). For an arbitrary $f \in C_c^\infty(G(\mathbf{A}))$, take a sequence of K-finite functions $\{f_n\}$ that approaches f. Define $I_\pi(f,\phi)$, $I_{\chi,s}(f,\phi)$ and $I_1'(f,\phi)$ as limits of $I_\pi(f_n,\phi)$, $I_{\chi,s}(f_n,\phi)$ and $I_1'(f_n,\phi)$. From the estimates in [A] and (24), and the dominance convergence theorem, the limits exist and the sums and integrals in (36) are again absolutely convergent. ∎

(3.3) We consider here the distribution

$$J(f') = \int_{\tilde{N}(F)\backslash\tilde{N}(\mathbf{A}))^2} K_{f'}(\tilde{n}_1^{-1}, \tilde{n}_2)\theta'(\tilde{n}_1 \cdot \tilde{n}_2)d\tilde{n}_1 d\tilde{n}_2 \tag{37}$$

for $f' \in C_c^\infty(G'(\mathbf{A}))$ an anti-genuine function. The situation here is simpler. As in (3.2)

$$K_{f'}(x,y) = \sum_\pi K_{\pi,f'}(x,y) + \sum_\chi K_{\chi,f'}(x,y),$$

where π is either a cuspidal representation or lies in the residue spectrum, and χ is the restriction of an idele class character onto $\mathbf{A}^{\times 3} = \{a^3 | a \in \mathbf{A}^\times\}$. Also

$$K_{\chi,f'}(x,y) = \int_{-i\infty}^{+i\infty} K_{\chi,s,f'}(x,y)ds.$$

Define $J_{\pi'}(f')$ and $J_{\chi',s}(f')$ by replacing $K_{f'}$ with the corresponding $K_{\pi',f'}(x,y)$ or $K_{\chi',s,f'}(x,y)$ in the expression (37). Using the estimates in [A], we see that

$$J(f') = \sum_{\pi'} J_{\pi'}(f') + \sum_{\chi} \int_{-i\infty}^{+i\infty} J_{\chi,s}(f')ds.$$

The sum and integral converge absolutely. We remark that $J_{\pi'}(f')$ is a nontrivial distribution iff π' has a nontrivial Whittaker model.

The integral (37) is known as a Kutznietsov trace. The unwinding of $J(f')$ is well known. Let

$$w = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad \text{and} \quad \tilde{w} = (w, 1).$$

Recall that an element g in $G(F)$ is identified with $(g,1)$ in $\tilde{G}(\mathbf{A})$. To unwind (37), we study the orbit of $\tilde{N} \times \tilde{N}$ on the group $\tilde{G}$ by the action

$$(g, \pm 1) \to \tilde{n}(t_1)^{-1}(g, \pm 1)\tilde{n}(t_2), (\tilde{n}(t_1), \tilde{n}(t_2)) \in \tilde{N} \times \tilde{N}.$$

An orbit for $g \in G(F)$ is called **relevant** if for any $(\tilde{n}(t_1), \tilde{n}(t_2)) \in \tilde{N} \times \tilde{N}(\mathbf{A})$ that fixes g, we have $\psi(t_1 - t_2) = 1$. Since the cocycle $\beta(g, n(t)) = 1$ for all t and g, the relevant orbits are the same as these in the linear case treated in [J-Y]. Thus the representatives for relevant orbits are given by ± 1 and wd_a with $a \in F^\times$. When $f' = \otimes f'_v$, similar to Proposition 2, we have

$$J(f') = \sum_{a \in F^\times} \prod J_v(a_v, f'_v) + \prod J_v^+(f'_v) + \prod J_v^-(f'_v) \tag{38}$$

where

$$J_v(a, f'_v) = \int_{(\tilde{N}(F_v))^2} f'_v(\tilde{n}_1 \cdot \tilde{w} \cdot d_a \cdot \tilde{n}_2)\theta'(\tilde{n}_1\tilde{n}_2)d\tilde{n}_1 d\tilde{n}_2 \tag{39}$$

and

$$J_v^{\pm}(f') = \int_{\tilde{N}(F_v)} f'_v(\pm\tilde{n})\theta'(\tilde{n})d\tilde{n}. \tag{40}$$

If $n = n(t)$, we fix the measure $d\tilde{n}$ to be dt.

4. Local integrals

Fix a local field F_v; we study the space of functions on $F_v^\times$: $\hat{I}_v = \{I_v(a, f_v * \phi_v)\}$ and $\hat{J}_v = \{J_v(a, f'_v)\}$. We will drop the reference to the local place v. We will show a matching between these two spaces of functions.

Let f, ϕ, f' be functions as in the previous paragraphs. We say the pair of functions (f, ϕ) matches the function f' if

$$I(-a, f * \phi) = |\sqrt{-3}|^{-1}|a|J(a, f') \tag{41}$$

and $I^\pm(f * \phi) = J^\pm(f')$. The main result of this section is:

THEOREM 1: *Given f', there is a matching pair (f, ϕ). When F is a non-archimedean field, the converse holds.*

(4.1) Let f and ϕ be smooth functions of compact support on $G(F)$ and F^2; then $f * \phi$ is a smooth function of compact support on F^2. Conversely, given ϕ, there exists f and ϕ' with $\phi = f * \phi'$. Thus $\hat{I}$ is the space of functions $\{I(a, \phi)\}$. Given ϕ, define

$$H_0(\phi)(a) = \int \phi(a, z)\psi\Big(\frac{z}{a}\Big)dz, \tag{42}$$

$$H^\pm(\phi)(a) = \int \phi(a, z)\psi\Big(\pm\frac{z^2}{a}\Big)dz. \tag{43}$$

Write $U_\epsilon(b)$ for the set $\{x||x-b| < \epsilon\}$ and Ω for the image of $F\backslash\{\pm 1\}$ under the map $z \to z^3 - 3z$. Let $\epsilon > 0$ be sufficiently small so that there is a smooth injective map $\delta : z \to t$ from $U_\epsilon(0)$ to F with $3z^2 = 3t^2 + t^3$. The space $\hat{I}$ can be described by the following:

PROPOSITION 4: *Given ϕ, there exist smooth functions ϕ' and $\phi^\pm$, where ϕ' is supported on $F \times \Omega$ and $\phi^\pm$ are supported on $F \times U_\epsilon(0)$, such that $I^\pm(\phi) = \phi^\pm(0,0)$ and*

$$I(-a, \phi) = |\sqrt{-3}|^{-1}[H_0(\phi')(a) + \psi(2/a)H^+(\phi^+)(a) + \psi(-2/a)H^-(\phi^-)(a)]. \tag{44}$$

Conversely, given $\phi', \phi^\pm$ as above, there is a ϕ satisfying the above equations.

Proof: Choose $\epsilon' > 0$ such that $\delta^{-1}(U_{\epsilon'}(0)) \subset U_{|\sqrt{-3}|^{-1}\epsilon}(0)$. By partition of unity, we may write ϕ as a sum of ϕ_i, $i = -1, 0, 1$, where ϕ_0 is supported on $F \times F\backslash\{\pm 1\}$ and $\phi_{\pm 1}$ are supported on $F \times U_{\epsilon'}(\pm 1)$ resp. Then $I^\pm(\phi) = \phi_{\pm 1}(0, \pm 1)$.

Recall

$$I(a, \phi_0) = \int \phi_0(a, t)\psi\Big(\frac{t^3 - 3t}{a}\Big)dt. \tag{45}$$

Over the support Ω_0 of ϕ_0 in variable t, there is no critical point of the polynomial $t^3 - 3t$. Thus Ω_0 can be covered by a finite union of open sets $U_{\epsilon_b}(b)$; over each set the map $t \to t^3 - 3t$ has a smooth inverse map. Write $\phi_0 = \sum_b \phi_b$, where ϕ_b is supported on $F \times U_{\epsilon_b}$. For each ϕ_b, after a change of variable $z = t^3 - 3t$, $I(-a, \phi_b)$ is of the form

$$\int \phi_b'(a,z)\psi\Big(\frac{z}{a}\Big)dz = H_0(\phi_b')$$

for some smooth function ϕ_b' of compact support on $F \times \Omega$. Let $\phi' = \sum_b |\sqrt{-3}|\phi_b'$; then $I(a, \phi_0) = |\sqrt{-3}|^{-1}H(\phi')(a)$.

Now we consider $I(a, \phi_1)$. Write $t = t' + 1$ in (45); we get

$$I(a,\phi_1) = \int_{U_\epsilon(0)} \phi_1(a, t'+1)\psi\Big(-\frac{2}{a}\Big)\psi\Big(\frac{t'^3 + 3t'^2}{a}\Big)dt'.$$

From the assumption on ϵ and ϵ', there is a smooth map $t' \to z$ from $U_{\epsilon'}(0)$ to $U_{|\sqrt{-3}|^{-1}\epsilon}(0)$, such that $3z^2 = 3t'^2 + t'^3$. Clearly $dz/dt' = 1$ at $z = 0$. Therefore the above integral is of the form

$$\int \phi_1'(a,z)\psi\Big(-\frac{2}{a}\Big)\psi\Big(\frac{3z^2}{a}\Big)dz$$

for some smooth function ϕ_1' supported on $F \times U_{\sqrt{-3}\epsilon}(0)$. Moreover, $\phi_1'(0,0) = \phi_1(0,1)$. Make a change of variable $z \to z/\sqrt{-3}$ and let $\phi^+(a,z) = \phi_1'(-a, z/\sqrt{-3})$; then $\phi^+(a,z)$ is supported on $F \times U_\epsilon(0)$ with

$$I(-a,\phi_1) = |\sqrt{-3}|^{-1}\psi(2/a)H^+(\phi^+)(a)$$

and $I^+(\phi) = \phi_1(0,1) = \phi^+(0,0)$.

Similarly, we can show that there is a ϕ^- with $I^-(\phi) = \phi^-(0,0)$ and $I(-a, \phi_{-1}) = |\sqrt{-3}|^{-1}\psi(-2/a)H^-(\phi^-)(a)$.

With our choice of $\phi', \phi^\pm$, the identities in the proposition are satisfied. Tracing the above argument backwards, we get the converse. ∎

(4.2) For $f' \in C_c^\infty(\tilde{G}(F))$, define

$$\Phi(a,y) = \int_F f'\Big(\tilde{n}(x)\cdot\Big(\begin{bmatrix} y^{-1} & \\ a & y \end{bmatrix}, e(y)\Big)\Big)\psi(x)dx,$$

where $e(y) = (y,a)$ if $v(a)$ does not divide 3 and $|3| = 1$; it is 1 otherwise. Given f', $\Phi(a,y)$ is a smooth function of compact support on $F \times F^\times$. When a is sufficiently close to 0, the support of Φ as a function of y lies in $F^\times$. Conversely, for any such Φ, there exists an antigenuine f' such that $\Phi(a,y)$ is obtained as above. One may verify that

$$J(a,f') = \int \Phi(a,y)\psi\Big(\frac{y+y^{-1}}{a}\Big)|a|^{-1}dy \tag{46}$$

and $J^{\pm}(f') = \Phi(0,\pm 1)$. Let Ω' be the image of $F^{\times}\backslash\{\pm 1\}$ under the map $z \to z + z^{-1}$. As in Proposition 4, one can find a sufficiently small $\epsilon > 0$ such that the following holds:

PROPOSITION 5: *Given f', there exist smooth functions Φ' and $\Phi^{\pm}$, where Φ' are supported on $F \times \Omega'$ and $\Phi^{\pm}$ is supported on $F \times U_{\epsilon}(0)$, such that $J^{\pm}(f') = \Phi^{\pm}(0,0)$ and*

$$J(a,f') = |a|^{-1}[H_0(\Phi')(a) + \psi(2/a)H^{+}(\Phi^{+})(a) + \psi(-2/a)H^{-}(\Phi^{-})(a)]. \tag{47}$$

Conversely, given $\Phi', \Phi^{\pm}$ as above, there is a f' satisfying the above equations.

Proof: Given $\Phi', \Phi^{\pm}$, we construct f'. Using the argument in the proof of Proposition 4 to equation (46), we can find a function f_1 with $J^{\pm}(f_1) = \Phi^{\pm}(0,0)$ and $J(a,f_1)$ equals the right-hand side of (47) when a is sufficiently close 0. Let $h(a)$ be the difference between the right-hand side of (47) and $J(a,f_1)$. Then $h(a)$ is a smooth function with compact support in $F^{\times}$. Let

$$\Phi(a,y) = \psi\Big(-\frac{y+y^{-1}}{a}\Big)h(a)C(y)|a|$$

where $C(y)$ is the characteristic function of a set on $F^{\times}$ with measure 1. Let f_2 be an antigenuine function corresponding to $\Phi(a,y)$ as above. Then $J(a,f_2) = h(a)$ and $J^{\pm}(f_2) = 0$. Let $f' = f_1 + f_2$, that is, the function we need.

The converse follows similarly. ∎

(4.3) *Proof of the Theorem:* From the remark in the beginning (4.1), we need only prove the matching between functions ϕ and f' instead of between (f,ϕ) and f'.

Start with f'; there correspond functions $\Phi', \Phi'^{\pm}$ as in Proposition 5. When the field is archimedean, $\Omega = \mathbf{C}$ and $\Omega' = \mathbf{C}\backslash\{\pm 2\}$. From Proposition 4, to these functions, there is a function ϕ satisfying the identities in the proposition with $\phi', \phi^{\pm}$ being replaced by $\Phi', \Phi'^{\pm}$. Compare the equalities (44) and (47); one sees immediately that $I(-a,\phi) = |\sqrt{-3}|^{-1}|a|J(a,f')$. The other identity $I^{\pm}(\phi) = J^{\pm}(f')$ also follows.

When the field is nonarchimedean, one can show that $H_0(\Phi')(a)$ is a smooth function on $F^{\times}$ which equals 0 when a is close to 0; thus it is compactly supported on $F^\times$. It is then possible to find a ϕ' compactly supported on $F^\times \times \Omega$ with $H_0(\phi') = H_0(\Phi')$. For example, let

$$\phi'(a,z) = \psi(-z/a)H_0(\Phi')(a)C(z)$$

where $C(z)$ is the characteristic function of a subset in Ω with measure 1. The above proof works again. Similarly, start with a ϕ; one can find a f' matching ϕ. ∎

5. Local integral: unramified case

In this section, we keep the notations and assumptions in §2. In particular we assume $|3| = 1$. In addition, assume ψ is of order 0. Let ϕ_0 be the characteristic function of the lattice R^2. We prove that if f and f' are matching Hecke functions under Proposition 1, then the pair (f, ϕ_0) match the function f' in the sense of (41).

(5.1) Let $\Phi_i(x)$ be the characteristic function of the set $|x| = q^i$. Define

$$I_m(a) = \int_{|t|\le q^m} \psi\Big(\frac{t^3-3t}{a}\Big)dt. \tag{48}$$

LEMMA 1: *When $m = 0$, $I(a, f_0 * \phi_0) = \sum_{i=-\infty}^{0} \Phi_i(a)I_0(a)$. When $m > 0$,*

$$I(a, f_m * \phi_0) = \sum_{i=0}^{2} \Phi_{3m-i}(a)I_m(a). \tag{49}$$

Proof: We first determine $f_m * \phi_0$. As ϕ_0 is fixed under the action of $\rho_\psi(Sp_2(R))$ and $s^3(k) \in Sp_2(R)$ for $k \in K$, we see that

$$\rho_\psi(s^3(k))\phi_0 = \phi_0, \quad k \in K.$$

If $m = 0$, then $f_0 * \phi_0 = \phi_0$ by (16) and the lemma follows. Now assume $m > 0$. Using the decomposition of $Kd_{\varpi^m}K$ in (8), we get

$$\begin{aligned} f_m * \phi_0 &= \int f_m(g^{-1})\rho_\psi(s^3(g))\phi_0 dg \\ &= \sum_{i=-m}^{m} h_i \end{aligned}$$

with

$$h_m(x,y) = \sum_{z\in R/P^{2m}} \rho_\psi(s^3(n(z)d_{\varpi^m}))\phi_0(x,y), \tag{50}$$

$$h_{-m}(x,y) = \rho_\psi(s^3(d_{\varpi^{-m}}))\phi_0(x,y), \tag{51}$$

$$h_i(x,y) = \sum_{z\in R^\times/P^{m+i}} \rho_\psi(s^3(n(\varpi^{i-m}z)d_{\varpi^i}))\phi_0(x,y), \tag{52}$$

for $-m < i < m$.

From the above expression for $f_m * \phi_0$, one has

$$I(a, f_m * \phi_0) = \sum_{i=-m}^{m} I(a, h_i). \tag{53}$$

We claim $I(a, h_i) \equiv 0$ when $i < m-1$.

From the definition, $h_{-m}(a,t) = q^{2m}\phi_0(a\varpi^{-3m}, t\varpi^{-m})$; the integral $I(a, h_{-m})$ is

$$q^{2m}\int_{|t|\le q^{-m}} \psi\Big(\frac{t^3-3t}{a}\Big)dt$$

when $|a| \le q^{-3m}$ and 0 otherwise. Let $T = t^3 - 3t$. The above integral becomes

$$q^{2m}\int_{|T|\le q^{-m}} \psi(T/a)dT,$$

which is apparently 0.

For $i > -m$, one may write $h_i(x,y)$ as an integral:

$$q^{m+i}\int \rho_\psi(s^3(n(\varpi^{i-m}z)d_{\varpi^i}))\phi_0(x,y)dz,$$

where the integral is over $|z| = 1$ if $i < m$ and $|z| \le 1$ if $i = m$. Observe that

$$\psi\Big(\frac{t^3-3t}{a}\Big)h_i(a,t) = \rho_\psi(s^3(n(t/a)))h_i(a,0)\psi\Big(-\frac{3t}{a}\Big).$$

Thus for $-m < i < m-1$,

$$I(a,h_i) = \int_{|z|=1} q^{m+i}\rho_\psi\left(s^3\left(n\left(\frac{t}{a}+\varpi^{i-m}z\right)d_{\varpi^i}\right)\right)\phi_0(a,0)\psi\Big(-\frac{3t}{a}\Big)dtdz. \tag{54}$$

Make a change of variable $t \to t - \varpi^{i-m}az$,

$$I(a,h_i) = \int_{|z|=1} q^{m+i}\rho_\psi\left(s^3\left(n\left(\frac{t}{a}\right)d_{\varpi^i}\right)\right)\phi_0(a,0)\psi\Big(-\frac{3t}{a}\Big)\psi(-\varpi^{i-m}z)dtdz. \tag{55}$$

The integration over z is 0 as $i < m-1$. We have proved our claim.

Now consider the integrals $I(a, h_{m-1})$ and $I(a, h_m)$. Equation (54) holds in the case $i = m-1$. Integrating over z in (55):

$$\begin{aligned} I(a, h_{m-1}) &= -\int q^{2m-2}\rho_\psi\left(s^3\left(n\left(\frac{t}{a}\right)d_{\varpi^{m-1}}\right)\right)\phi_0(a,0)\psi\Big(-\frac{3t}{a}\Big)dt \\ &= -\int \psi\Big(\frac{t^3-3t}{a}\Big)\phi_0(a\varpi^{3m-3}, t\varpi^{m-1})dt \\ &= \sum_{j=-\infty}^{3m-3} -\Phi_j(a)I_{m-1}(a). \end{aligned}$$

When $i = m$, equation (54) holds if we replace the domain of integration by $|z| \leq 1$. Proceed as above; one gets

$$\begin{aligned} I(a, h_m) &= \int \psi\Big(\frac{t^3-3t}{a}\Big)\phi_0(a\varpi^{3m}, t\varpi^{m})dt \\ &= \sum_{j=-\infty}^{3m} \Phi_j(a)I_m(a). \end{aligned}$$

As $I(a, f_m * \phi) = I(a, h_{m-1}) + I(a, h_m)$, to prove the lemma, we only need to show that when $|a| \leq q^{3m-3}$, $I_{m-1}(a) = I_m(a)$, i.e.

$$\int_{|t|=q^m} \psi\Big(\frac{t^3-3t}{a}\Big)dt = 0. \tag{56}$$

The set $|t| = q^m$ is a disjoint union of cosets of $P^{-v(a)+2m-1}$. For each of these cosets with a representative t_0, with $|t_0| = q^m$, the contribution to $I(a, f_m, \phi_0)$ is

$$\int_{|v|\leq|a|q^{1-2m}} \psi\left(\frac{t_0^3 - 3t_0 + 3t_0^2 v}{a}\right)\psi\left(\frac{3t_0v^2+v^3-3v}{a}\right)dv$$

Since

$$\psi\left(\frac{3t_0v^2+v^3-3v}{a}\right) = 1$$

over the domain and $|3t_0^2/a| = |a|^{-1}q^{2m}$, the above integration equals 0. We have proved equation (56), thus the lemma. ∎

(5.2) We study the integrals $I_m(a)$ when $m > 0$ and $|a| = q^{3m}, q^{3m-1}$ or q^{3m-2}. If $|x| = q^{-1}$, χ is a nontrivial cubic character on $F^\times$; let $G(x,\chi)$ be the Gaussian sum

$$\int_{|y|=1} \psi(x^{-1}y)\chi(y)dy.$$

When $|a| = q^{3m}$, the integrand in (48) is 1, thus $I_m(a) = q^m$.

When $|a| = q^{3m-1}$ or q^{3m-2},

$$\begin{aligned} I_m(a) &= \int_{|t|\le q^m} \psi\Big(\frac{t^3}{a}\Big)dt \\ &= \sum_{v\in R/P} \int_{|u|\le q^{m-1}} \psi\Big(\frac{(u+v\varpi^{-m})^3}{a}\Big)du \\ &= \sum_{v\in R/P} \int_{|u|\le q^{m-1}} \psi\Big(\frac{v^3\varpi^{-3m}+3v^2\varpi^{-2m}u}{a}\Big)du \end{aligned}$$

as

$$\psi\left(\frac{3v\varpi^{-m}u^2+u^3}{a}\right) = 1$$

over the domain. When $|a| = q^{3m-2}$, the integration over u is nonzero only when $v = 0$, thus

$$I_m(a) = \int_{|u|\le q^{m-1}} du = q^{m-1}.$$

When $|a| = q^{3m-1}$, we have

$$\psi\Big(\frac{3v^2\varpi^{-2m}u}{a}\Big) = 1$$

over the domain. Thus

$$\begin{aligned} I_m(a) =& q^{m-1} \sum_{v\in R/P} \psi\left(\frac{v^3\varpi^{-3m}}{a}\right) \\ =& q^{m-1} \sum_{v\in R/P} \sum_{i=0}^{2} \psi\left(\frac{v\varpi^{-3m}}{a}\right) \chi^i(v), \end{aligned}$$

where $\chi(v)$ is any nontrivial cubic character on $R^\times/P$ and $\chi(0) = 0$. Clearly in the above sum, the part $i = 0$ contributes 0. Use the Gaussian sum notation; $I_m(a) = q^m(G(a\varpi^{3m},\chi) + G(a\varpi^{3m},\chi^2))$.

In conclusion, from Lemma 1 we get

Proposition 6: *When $m > 0$, $I(a, f_m * \phi_0)$ equals*

(57) $q^m\Phi_{3m}(a) + q^m(G(a\varpi^{3m},\chi) + G(a\varpi^{3m},\chi^2))\Phi_{3m-1}(a) + q^{m-1}\Phi_{3m-2}(a)$.

(5.3) When $m = 0$, from Lemma 1, $I(a, f_0 * \phi_0) = I_0(a)$ when $|a| \le 1$ and 0 otherwise. We now compute $I_0(a)$ when $|a| \le 1$.

When $|a| = 1$, $I_0(a) = 1$.

When $|a| = q^{-1}$, one can write $I_0(a)$ in the following form:

$$I_0(a) = \sum_{v \in R/P} q^{-1} \psi\left(\frac{v^3 - 3v}{a}\right). \tag{58}$$

When $|a| = q^{-2l}$ with $l > 0$ or q^{-2l+1} with $l > 1$, we separate the set $|t| \leq 1$ into P^l cosets $S(t_0)$ with representatives t_0. One can show that only the cosets $S(t_0)$ with $|3t_0^2 - 3| \leq |a|q^l$ give nonzero contributions to $I_0(a)$. When $|a| = q^{-2l}$, there are only two cosets $S(\pm 1)$; then

$$I_0(a) = q^{-l}[\psi(2/a) + \psi(-2/a)]. \tag{59}$$

When $|a| = q^{-2l+1}$, the cosets are of the form $S(\pm 1 + v\varpi^{l-1})$ with $v \in R/P$; thus

$$I_0(a) = q^{-l}\left[\psi\left(-\frac{2}{a}\right) \sum_{v \in R/P} \psi\left(\frac{3v^2\varpi^{2l-2}}{a}\right) + \psi\left(\frac{2}{a}\right) \sum_{v \in R/P} \psi\left(\frac{-3v^2\varpi^{2l-2}}{a}\right)\right]. \tag{60}$$

(5.4) For f'_m a function defined in §2.3, define

$$J_m(a) = \int f'_m(\tilde{n}(x) \cdot \tilde{w} \cdot (d_a, 1))\psi(x)dx. \tag{61}$$

This is the contribution to the integral (39) from the set with $\tilde{n}_2 \in \tilde{N} \cap K$. One may verify that

$$J(a, f'_m) = J_m(a) + \int_{|y|>1} J_m(ay)\psi(y + a^{-2}y^{-1})(a, y)dy. \tag{62}$$

We first determine $J_m(a)$.

Lemma 2: *When $m = 0$, $J_0(a) = \Phi_0(a)$. When $m > 0$,*

$$J_m(a) = \Phi_{3m}(a) + q\Phi_{3m-1}(a)G(a\varpi^{3m}, \chi_a),$$

where $\chi_a(x) = (x, a)$ is a nontrivial cubic character on $F^\times$.

Proof: The assertion on $J_0(a)$ is clear. Assume $m > 0$. Over the support of f'_m, in the integral for $J_m(a)$, we have either of the three cases:

(1) $|a| = q^{3m}$ and $|x| \leq 1$.
(2) $|a| = q^{-3m}$ and $|x| \leq q^{6m}$.
(3) $|a| = q^i$ with $-3m < i < 3m$ and $|x| = q^{3m-i}$.

The integral over x will be 0 unless we are in case (3) with $|a| = q^{3m-1}$ or in case (1). Thus when $|a| = q^{3m}$, $J_m(a) = 1$. When $|a| = q^{3m-1}$,

$$J_m(a) = \int_{|x|=q} f'_m\left(\begin{bmatrix} ax & -a^{-1} \\ a & \end{bmatrix}, 1\right)\psi(x)dx$$
$$= \int_{|x|=q} \psi(x)(x,a)dx.$$

Make a change of variable $x \to xa^{-1}\varpi^{-3m}$; as $(a,a) = 1$, we get

$$J_m(a) = qG(a\varpi^{3m}, \chi_a).$$

This proves the lemma when $m > 0$. ∎

(5.5) Let $G_a = G(a\varpi^{3m}, \chi_a)$ when $|a| = q^{3m-1}$ and 1 otherwise. Define

$$KJ_l(a) = \int_{|y|>1} \Phi_l(ay)G_{ay}\psi(y + a^{-2}y^{-1})(a,y)dy \tag{63}$$

for $l \geq 0$ of the form $3m, 3m-1$. From Lemma 2, $J(a, f'_0) = \Phi_0(a) + KJ_0(a)$ and, when $m > 0$,

$$J(a, f'_m) = \Phi_{3m}(a) + q\Phi_{3m-1}(a)G(a\varpi^{3m}, \chi_a) + KJ_{3m}(a) + qKJ_{3m-1}(a). \tag{64}$$

We compute $KJ_l(a)$ for $l > 1$. Make a change of variable $y \to a^{-1}y$ in (63):

$$KJ_l(a) = \int_{q^l=|y|>|a|} G_y\psi\left(\frac{y+y^{-1}}{a}\right)(a,y)|a|^{-1}dy. \tag{65}$$

Let $z = y + y^{-1}$; then when $l > 1$, over the domain of (65), $G_z = G_y$ and $(a,y) = (a,z)$; the integral becomes

$$\int_{q^l=|z|>|a|} G_z\psi(\frac{z}{a})(a,z)|a|^{-1}dz. \tag{66}$$

Therefore when $l = 3m > 0$,

$$KJ_{3m}(a) = \int_{q^{3m}=|z|>|a|} \psi\left(\frac{z}{a}\right)(a,z)|a|^{-1}dz.$$

This integral is nonzero only when $|a| = q^{l-1}$; as $(a,z) = (z,a)^2$, with the above notations,

$$KJ_{3m}(a) = q\Phi_{3m-1}(a)G(a\varpi^{3m}, \chi_a^2). \tag{67}$$

When $l = 3m-1$, the integral (66) equals

$$KJ_{3m-1}(a) = \int_{q^{3m-1}=|z|>|a|}\int_{|v|=1} \psi\left(\frac{v}{z\varpi^{3m}}\right)(v,z)\psi\left(\frac{z}{a}\right)(a,z)|a|^{-1}dzdv.$$

Change $v \to vz\varpi^{3m}/a$:

$$KJ_{3m-1}(a) = q^{-1}\int_{q^{3m-1}=|z|>|a|}\int_{|v|=q|a|} \psi\left(\frac{v+z}{a}\right)|a|^{-2}(v,z)dzdv.$$

For the integration over z to be nonzero, $|a|$ must be q^{3m-2};

$$KJ_{3m-1}(a) = \Phi_{3m-2}(a)q^{3-6m}\int_{|v|=|z|=q^{3m-1}} \psi\left(\frac{v+z}{a}\right)(v,z)dvdz,$$

which equals $\Phi_{3m-2}(a)$.

In conclusion,

PROPOSITION 7: *When $m > 0$, $J(a, f'_m)$ equals*

(68) $\Phi_{3m}(a) + q\Phi_{3m-1}(a)G(a\varpi^{3m}, \chi_a) + q\Phi_{3m-1}(a)G(a\varpi^{3m}, \chi_a^2) + q\Phi_{3m-2}(a).$

(5.6) Comparing Proposition 6 and Proposition 7, we get

THEOREM 2: *When $m > 0$, the function $q^{-2m}f'_m$ matches the pair (f_m, ϕ_0).*

Proof: As we can certainly take χ to be χ_{-a} in Proposition 6, we get the equality

$$I(a, f_m * \phi_0) = |a|J(-a, q^{-2m}f'_m).$$

The equalities between $I^{\pm}(f_m * \phi_0)$ and $J^{\pm}(q^{-2m}f'_m)$ hold, as in fact they all equal 0. ∎

(5.7) We now prove the corresponding result in the case when $m = 0$.

THEOREM 3: *The function f'_0 matches the pair (f_0, ϕ_0).*

Proof: One verifies that $I^{\pm}(f_0 * \phi_0) = I^{\pm}(\phi_0) = 1$ and $J^{\pm}(f'_0) = 1$. We now show the identity

(69) $$I(a, f_0 * \phi_0) = J(-a, f'_0).$$

Recall $J(a, f'_0) = \Phi_0(a) + KJ_0(a)$, with

$$KJ_0(a) = \int_{|y|=1>|a|} \psi\left(\frac{y+y^{-1}}{a}\right)(a,y)|a|^{-1}dy.$$

Compare with (5.3); the identity (69) holds when $|a| \geq 1$. When $|a| < 1$, it will follow from

$$I_0(a) = |a| K J_0(-a). \tag{70}$$

When $|a| = q^{-1}$,

$$KJ_0(a) = \sum_{x \in R^\times / P} \psi\left(\frac{x + x^{-1}}{a}\right) \chi(x), \tag{71}$$

where $\chi(x) = (a, x)$ is a nontrivial cubic character on $R^\times/P$. The identity (70) follows from the formula in [D-I] which compares (58) and (71).

When $|a| = q^{-2l}$ with $l > 0$ or q^{-2l+1} with $l > 1$, we separate the set $|y| = 1$ into P^l cosets $S(y_0)$ with representatives y_0. One can show that only the cosets $S(y_0)$ with $|y_0^2 - 1| \leq |a| q^l$ give nonzero contributions to J_0'. When $|a| = q^{-2l}$, there are only two such cosets $S(\pm 1)$; then

$$KJ_0(a) = q^{-l} |a|^{-1} [\psi(2/a) + \psi(-2/a)]. \tag{72}$$

When $|a| = q^{-2l+1}$, the cosets are of the form $S(\pm 1 + v\varpi^{l-1})$ with $v \in R/P$; thus

$$KJ_0(a) = q^{-l}|a|^{-1}\left[\psi\left(\frac{2}{a}\right) \sum_{v \in R/P} \psi\left(\frac{v^2 \varpi^{2l-2}}{a}\right) + \psi\left(-\frac{2}{a}\right) \sum_{v \in R/P} \psi\left(-\frac{v^2 \varpi^{2l-2}}{a}\right)\right]. \tag{73}$$

Compare the expressions (72),(73) with (59),(60); in either case, (70) holds as -3 is a square. ∎

6. The comparison

(6.1) Back to the global situation. From the local results in Theorems 1, 2 and 3, we get

THEOREM 4: *Let $f = \otimes f_v, \phi = \otimes \phi_v$ and $f' = \otimes f_v'$ be functions smooth of compact support (f' being antigenuine). If for any v, (f_v, ϕ_v) and f_v' match, then*

$$I(f, \phi) = J(f'). \tag{74}$$

For any f' as above there exists (f, ϕ) that matches f' over all places. Moreover, at almost all finite places v, $f_v' \in H_v'$ is a spherical function, one can choose ϕ_v to be the characteristic function of R_v^2, and f_v to be the image of f_v' under the isomorphism of Hecke algebras $H_v' \to H_v$.

Proof: The first assertion follows from equations (17) and (38), the matching condition and the fact that $\prod |a|_v = 1$ when $a \in F^\times$.

Given $f' = \otimes f'_v$, at each place v, we can choose (f_v, ϕ_v) so that it matches f'_v. At almost all finite places v, $f'_v = f'_{0,v}$, and by Theorem 3, we may let f_v and ϕ_v be the characteristic functions of K_v and R_v^2, respectively. Then $(f = \otimes f_v, \phi = \otimes \phi_v)$ matches f' at all places.

The last assertion follows from Theorems 2 and 3. ∎

In §3, we decompose the distributions $I(f,\phi)$ and $J(f')$ into a sum of discrete parts and continuous parts. From a standard argument ([La] §11), it follows from Theorem 4 that when (f,ϕ) and f' match, then the continuous part of $I(f,\phi)$ and $J(f')$ are equal, i.e.

$$\sum_\pi I_\pi(f,\phi) + I''(f,\phi) = \sum_{\pi'} J_{\pi'}(f'), \tag{75}$$

$$\sum_\chi \int_{-\infty}^{+\infty} \frac{1}{4\pi i} I_{s,\chi}(f,\phi)ds = \sum_\chi \int_{-\infty}^{+\infty} J_{s,\chi}(f')ds. \tag{76}$$

We will exploit equation (75).

(6.2) Let S be a finite set of places containing all archimedean places and places where $|3| < 1$. For any $v \notin S$, let $\pi_v(|\cdot|^{s_v})$ be $\rho_v(|\cdot|^{s_v})$ (§2) if $s_v \neq \pm 1$, and be the unique unramified quotient of $\rho_v(|\cdot|)$ when $s_v = \pm 1$. For each set of $\{s_v | v \notin S\}$, we define an L-packet $\Pi(\{s_v\})$ of cuspidal representations on $G(\mathbf{A})$:

$$\{\pi = \otimes \pi_v | \pi_v = \pi_v(|\cdot|^{s_v}), v \notin S\}.$$

We say an L-packet Π is Θ-distinguished if there exists $\pi \in \Pi$, $f \in \pi$ such that, for some ϕ,

$$P(\phi, f) = \int_{G(F)\backslash G(\mathbf{A})} \Theta_\psi^\phi(s^3(g))f(g)dg \neq 0. \tag{77}$$

Meanwhile, let $\pi'_v(|\cdot|^{s_v})$ be $\tilde{\rho}(|\cdot|^{s_v})$ if $s_v \neq \pm\frac{1}{3}$, and be the unique unramified quotient of $\tilde{\rho}_v(|\cdot|^{1/3})$ when $s_v = \pm\frac{1}{3}$ ([Ar]). Define an L-packet $\Pi'(\{s_v\})$ of genuine cuspidal representations on $\tilde{G}(\mathbf{A})$:

$$\{\pi' = \otimes \pi'_v | \pi'_v = \pi'_v(|\cdot|^{s_v}), v \notin S\}.$$

We say the packet Π' is **generic** if there is a cuspidal representation in the packet which has a Whittaker model with respect to ψ.

We say Π' is CAP if $s_v = \pm\frac{1}{3}$ for all $v \notin S$. We expect that there is no Π' that is CAP. Assume this is the case, for any set S as above:

THEOREM 5: *There is a bijection between generic L-packets on $\tilde{G}(\mathbf{A})$ and Θ-distinguished L-packets on $G(\mathbf{A})$. The bijection is given by $\Pi'(\{s_v\}) \leftrightarrow \Pi(\{3s_v\})$.*

It is shown in [G-R-S] that if the L-packet $\Pi(\{3s_v\})$ is Θ-distinguished, then $\Pi'(\{s_v\})$ is nonempty. We now give a proof of the converse using the identity (75).

Proof: Denote by H'^S and H^S the product of Hecke algebras H'_v and H_v for $v \notin S$. Let $f' = \otimes f'_v$ be an antigenuine function chosen as follows: If f'^S is the tensor product of the functions f'_v for v not in S, then f'^S is an element of H'^S; if $v \in S$, let f'_v be the convolution of an antigenuine function $h_v(g)$ with $\overline{h_v(g^{-1})}$. Given f', let f'_S be the product of f'_v if $v \in S$, and $f'_{0,v}$ if $v \notin S$. Then $K_{\pi',f'}(x,y) \equiv 0$ unless $\pi' \in \Pi'(\{s_v\})$ for some set $\{s_v | v \notin S\}$, in which case

$$K_{\pi',f'}(x,y) = f'^{S\wedge}(\pi')K_{\pi',f'_S}(x,y),$$

where

$$f'^S \to f'^{S\wedge}(\pi') = \prod_{v \notin S} f'^{\wedge}_v(s_v)$$

is the character of the Hecke algebra H'^S corresponding to $\Pi'(\{s_v\})$. Thus

$$\sum_{\pi'} J_{\pi'}(f') = \sum_{\{s_v\}} \sum_{\pi' \in \Pi'(\{s_v\})} f'^{S\wedge}(\pi')J_{\pi'}(f'_S) + J_{res}(f'), \tag{78}$$

where $J_{res}(f')$ is the contribution to $J(f')$ from the residue spectrum of $\tilde{G}(\mathbf{A})$. It follows from the theory of Eisenstein series that there is one representation lying in the residue spectrum; its local components at finite places are $\pi'_v(|\cdot|^{1/3})$ [K-P]. Thus

$$J_{res}(f') = \prod_{v \notin S} f'^{\wedge}_v(1/3)J_{res}(f'_S).$$

When $v \notin S$, let ϕ_v be the characteristic function of R_v^2 and f_v be the image of f'_v under the isomorphism between Hecke algebras H_v and H'_v. For $v \in S$, pick (f_v, ϕ_v) that matches f'. Let $\phi = \otimes\phi_v$ and $f = \otimes f_v$. Then similar to expression (78), we have

$$\begin{aligned}\sum_{\pi} I_\pi(f,\phi) + I''_1(f,\phi) = &\sum_{\{s_v\}} \sum_{\pi \in \Pi(\{s_v\})} f^{S\wedge}(\pi)I_\pi(f_S,\phi) \\ &+ \prod_{v \notin S} f^{\wedge}_v(1)I''_1(f_S,\phi),\end{aligned} \tag{79}$$

where f_S is the product of $f_{0,v}$, $v \notin S$ and f_v, $v \in S$; and

$$f^S \to f^{S\wedge}(\pi) = \prod_{v \notin S} f^{\wedge}_v(s_v)$$

is the character of the Hecke algebra H^S. With the isomorphism in Proposition 1,

$$f'^S \to f^S \to f^{S\wedge}(\pi)$$

is the character of the Hecke algebra H'^S. Thus both sides of equation (75) can be expressed as infinite sums of characters of H'^S. With the linear independence of characters, as $f_v^\wedge(s) = f'^\wedge_v(s/3)$, we get for each $\{s_v\}$ (except when $s_v \equiv 1, v \notin S$, the CAP case)

$$\sum_{\pi\in\Pi(\{s_v\})} I_\pi(f_S,\phi) = \sum_{\pi'\in\Pi'(\{s_v/3\})} J_{\pi'}(f'_S). \tag{80}$$

As f'_S is a convolution of a function $h(g)$ with $\overline{h(g^{-1})}$, the function K_{π',f'_S} has the form

$$\sum_{\varphi_i} \pi'(h)\varphi_i(x)[\overline{\pi'(h)\varphi_i(y)}].$$

Then

$$J_{\pi'}(f'_S) = \sum_{\varphi_i} \left| \int_{\tilde N(F)\backslash \tilde N(\mathbf{A})} \pi'(h)\varphi_i(\tilde n)\theta'(\tilde n)d\tilde n \right|^2.$$

In particular, $J_{\pi'}(f'_S)$ is a positive number or 0. If the L-packet $\Pi(\{s_v/3\})$ is nonempty and generic, then one can choose a f' with $J_{\pi'}(f'_S)$ nonzero for one π' in the packet. The sum on the right-hand side of (80) is then nonzero; there exists at least one π in the L-packet $\Pi(\{s_v\})$ with $I_\pi(f_S,\phi) \neq 0$. From the expression for $I_\pi(f,\phi)$ in (28), we see that π is Θ-distinguished. ∎

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